106B Discussion: 4/21

The Final Discussion!

Today's Agenda:

1. Optimal Control
2. Dynamic Programming & OC
3. RL

Optimal Control:

=> MPC, CLF, CBF . . .

=> Solve optim. to get best input!!

- Want to develop a framework for solving OC probs!!

Class of O.C. Problems:

=> Some DT sys:

$$\boxed{x_{k+1} = f(x_k, u_k)}$$

(State vec. x_k; Input vec. u_k)

=> for some finite time (N), want to identify the BEST inputs to the sys!!

=> Optimal SEQUENCE of inputs:

$$\boxed{u_0, u_1, \ldots, u_{N-1}}$$

=> General type of cost func:

$$J = L_f(x_N) + \sum_{k=0}^{N-1} L(x_k, u_k)$$

$L_f(x_N)$: "TERMINAL COST"; $L(x_k, u_k)$: "STAGE COST"

=> Cost assoc. w/ states along the way & inputs!!

=> We want to solve:

$$u_0^*, u_1^*, \ldots, u_{N-1}^* = \arg\min_{u_0, u_1, \ldots, u_{N-1}} L_f(x_N) + \sum_{k=0}^{N-1} L(x_k, u_k)$$

=> REALLY HARD TO SOLVE!!

- Breaking it up into SMALLER SOLVABLE PROBS!!

Dynamic Programming:

"Optimal Cost-to-go": Imagine we've ALREADY taken i optimal steps!!

=> Remaining O.C. from $i \to N$

$$J_i^\circ = \min_{u_i \ldots u_{N-1}} L_f(x_N) + \sum_{k=i}^{N-1} L(x_k, u_k) \quad \text{optim. for } u_i$$

$$= \min_{u_i \ldots u_{N-1}} L_f(x_N) + L(x_i, u_i) + \sum_{k=i+1}^{N-1} L(x_k, u_k)$$

$$= \min_{u_i \ldots u_{N-1}} L(x_i, u_i) + \left(L_f(x_N) + \sum_{k=i+1}^{N-1} L(x_k, u_k) \right)$$

$$J_i^\circ = \min_{u_i \ldots u_{N-1}} L(x_i, u_i) + J_{i+1}^\circ(x_{i+1})$$

=> $x_{i+1} = f(x_i, u_i)$

$$\boxed{J_i^\circ = \min_{u_i} L(x_i, u_i) + J_{i+1}^\circ(f(x_i, u_i))}$$

☆ BELLMAN EQN! ☆

=> "Sys is noisy" $x_{k+1} = f(x_k, u_k, w_k)$ (w_k: "disturbance")

=> Dynamic Programming:

=> $J_N^\circ = L_f(x_N)$

$$J_{N-1}^\circ = \min_{u_{N-1}} L(x_{N-1}, u_{N-1}) + L_f(f(x_{N-1}, u_{N-1}))$$

: $u_{N-1}(x_{N-1})$

J_{N-2}° : . . .

=> Entire optimal input seq!!: $u_0, \ldots, u_{N-1}$

Example:

Consider the scalar DT sys:

$$\boxed{x_{k+1} = a x_k + b u_k} \quad x_k, u_k \in \mathbb{R}$$

=> Want to MIN:

$$\boxed{J = x_N^2 + \sum_{k=0}^{N-1} (x_k^2 + u_k^2)}$$

=> Figure out u_{N-1} w/ DP.

=> Try & identify term. $L_f(x_N)$, Stage $L(x_k, u_k)$

$$L_f(x_N) = x_N^2, \quad L(x_k, u_k) = x_k^2 + u_k^2$$

=> Start w/ our term. cost!:

$$J_N^\circ = L_f(x_N) = x_N^2$$

↓ go one step back!

$$J_{N-1}^\circ = \min_{u_{N-1}} L(x_{N-1}, u_{N-1}) + J_N^\circ(x_N) \quad \text{Bellman eqn!!}$$

$$= \min_{u_{N-1}} x_{N-1}^2 + u_{N-1}^2 + x_N^2 \quad \text{write in terms of } u_{N-1}$$

$$= \min_{u_{N-1}} x_{N-1}^2 + u_{N-1}^2 + (a x_{N-1} + b u_{N-1})^2$$

$$J_{N-1}^\circ = \min_{u_{N-1}} x_{N-1}^2 + u_{N-1}^2 + a^2 x_{N-1}^2 + 2ab\, x_{N-1} u_{N-1} + b^2 u_{N-1}^2$$

Min wrt u_{N-1}??

$$\frac{\partial J_{N-1}^\circ}{\partial u_{N-1}} = 2u_{N-1} + 2ab\, x_{N-1} + 2b^2 u_{N-1} = 0$$

Solve for u_{N-1}

$$u_{N-1}(1 + b^2) + ab\, x_{N-1} = 0$$

$$\boxed{u_{N-1} = -(1+b^2)^{-1} ab\, x_{N-1}}$$

=> Just state feedback!!

=> LQR Controller!!