Today's Agenda:

1) PFAFFIAN CONSTR
2) EQUIV. CONTROL SYS
3) LIE BRACKETS!!

PFAFFIAN CONSTRAINTS:

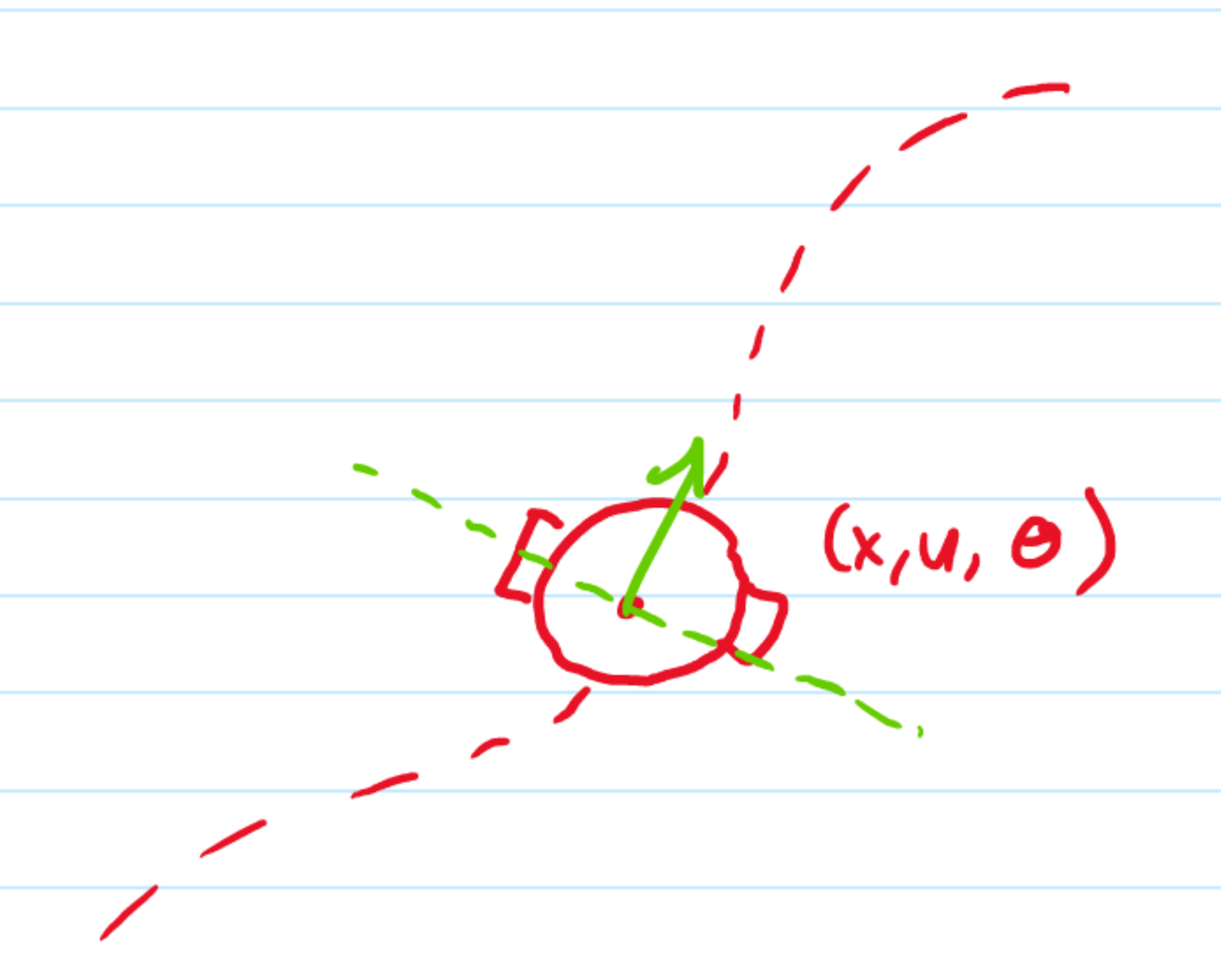
⇒ In 106A, talked about LAGRANGIAN DYNAMICS!
⇒ gen. coords $q_1, q_2, \dots, q_n$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

} 2^{nd} order diff. eqs! $\ddot{q}_1, \ddot{q}_2, \dots, \ddot{q}_n$

⇒ SIMPLER REPR. of kinematics!

Turtlebot:



Constraints:

1) Constraint on $\dot{x}, \dot{y}, \dot{\theta}$
2) Constr. on x, y, θ!
3) Turtlebot can only travel in forward/backward dir!! (not side to side!)

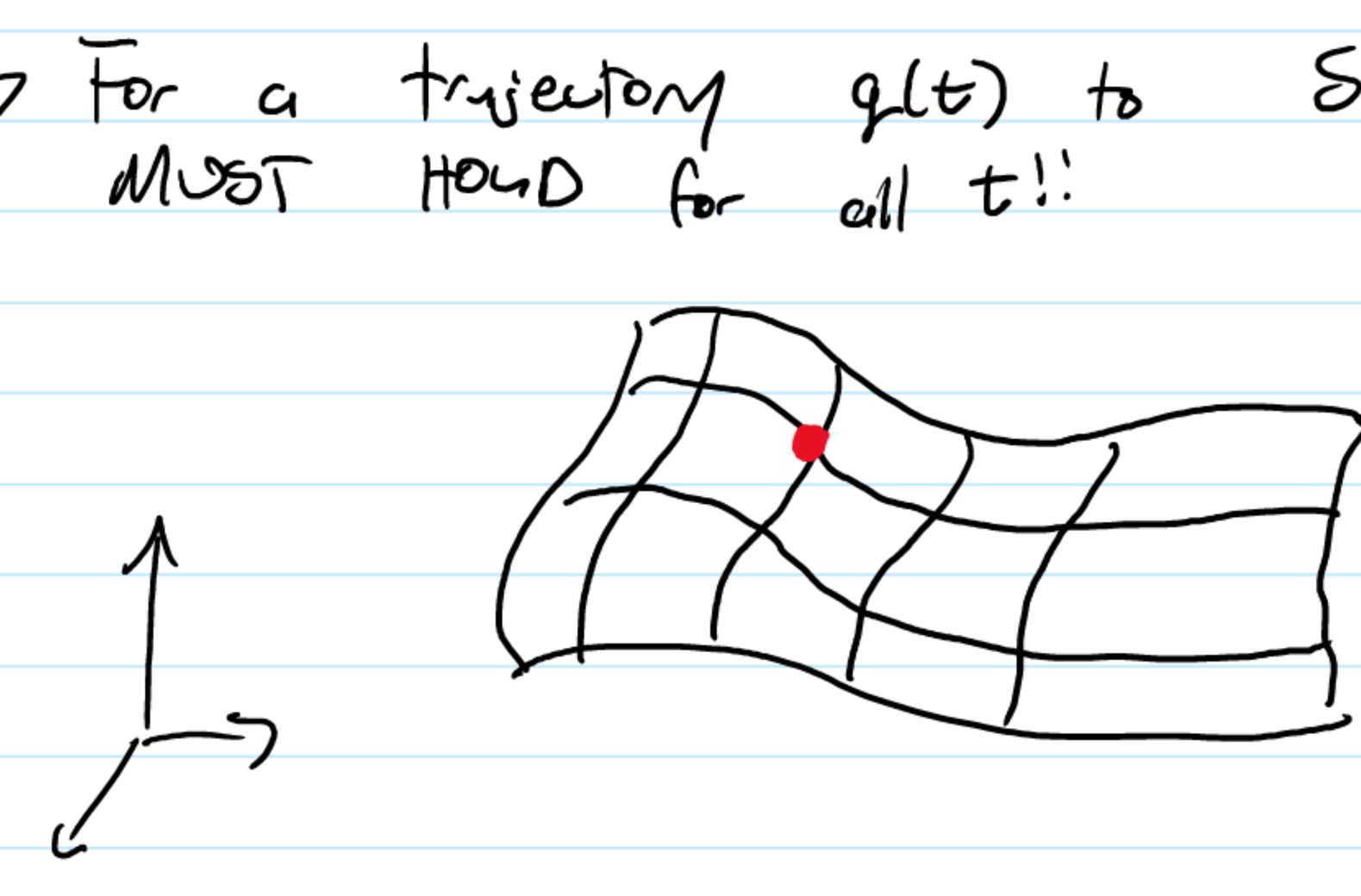
PFAFFIAN CONSTR: "convention" for writing constraints on motion!

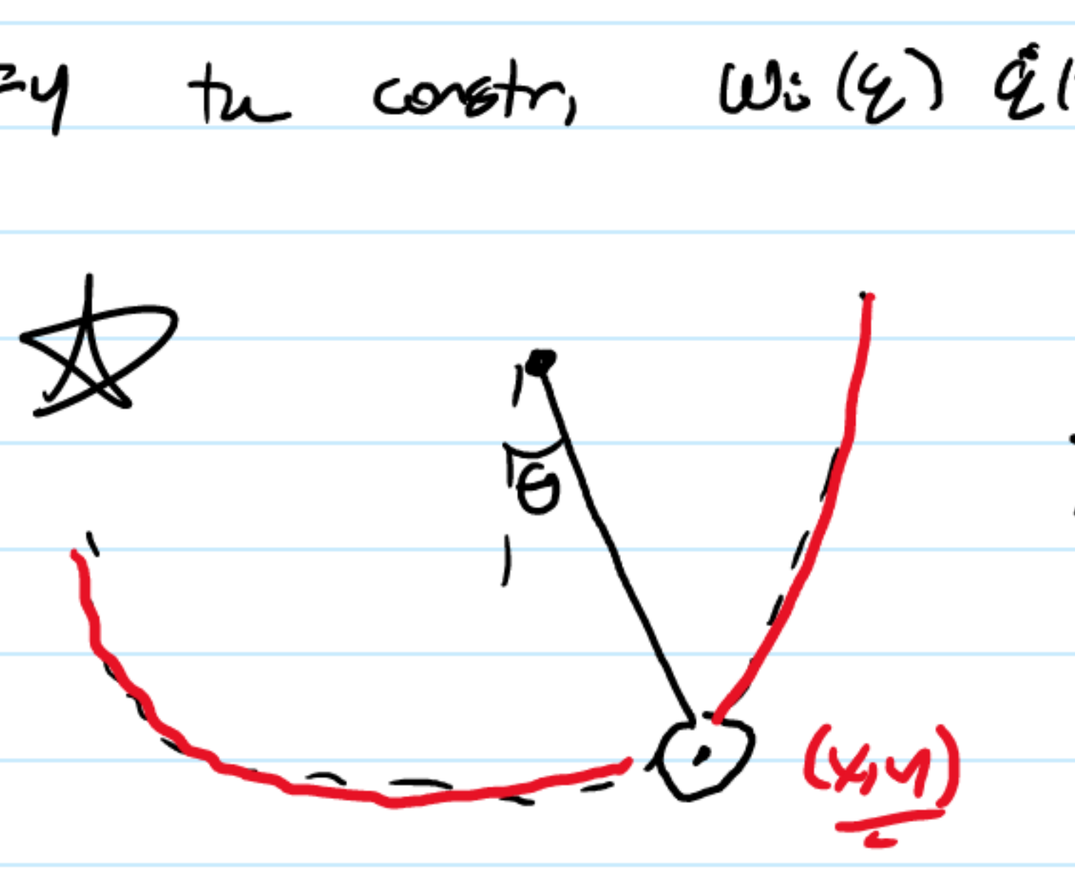
⇒ $q_1, q_2, \dots, q_n$ } gcs.

☆ $\boxed{\omega_i(q)\,\dot{q} = 0}$ ☆ $[\omega_{i1}(q) \dots \omega_{in}(q)]\begin{bmatrix}\dot{q}_1\\ \vdots \\ \dot{q}_n\end{bmatrix} = 0$

$\omega_i(q)$: function of g.cs, q
$\dot{q}$: imposes constraint on VELOCITIES of gen. coords!!

⇒ For a trajectory $q(t)$ to SATISFY the constr, $\omega_i(q)\,\dot{q}(t) = 0$ MUST HOLD for all t!!





⇒ $\omega_i(q)\,\dot{q} = 0$ } DIRECTLY constrains VELOCITIES!!
⇒ Q: does it dir. constrain our positions??
- Could directly impact what q we can reach!! } Not always true!!
- NOT all Pfaffian constr. DIRECTLY constr. pos!!

CLASSIFY THESE CONSTR:

- A constr. $\omega_i(q)\,\dot{q} = 0$ is said to be INTEGRABLE if it is EQUIV. to $h(q) = 0$

☆ $(\omega_i(q))\,\dot{q} = 0 \iff \boxed{h(q) = 0}$ ☆
JUST DEPENDS ON POS!!

⇒ Integrable → DIRECTLY constr. pos!!

What happens when we have a SYSTEM of constr??

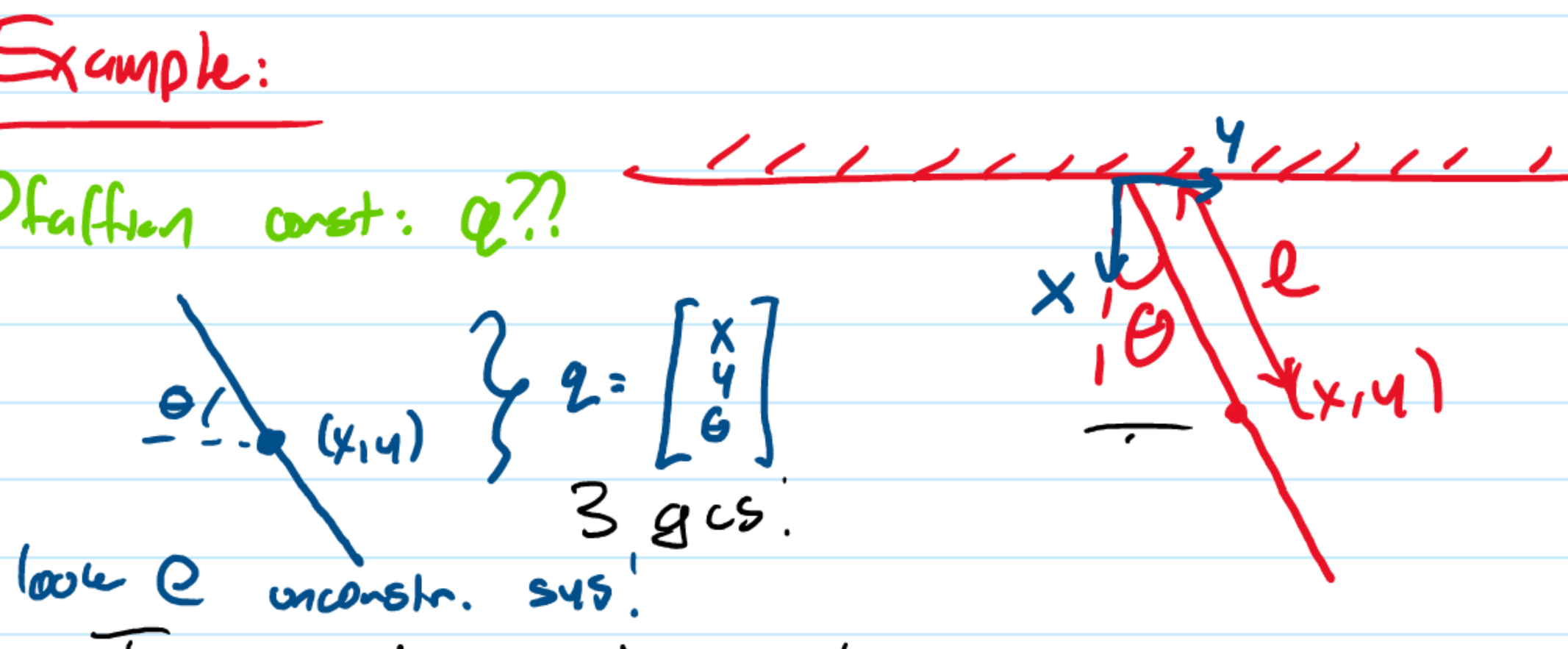
⇒ k constr: $\omega_1(q)\,\dot{q} = 0 \quad \dots \quad \omega_k(q)\,\dot{q} = 0$

⇒ If ALL k constr. are integrable ($\omega_i(q)\,\dot{q} = 0 \iff h_i(q) = 0$) then we say the SET of constr. is HOLONOMIC!!
- If NO constr. is integrable, we say the set of constr. is NONHOLONOMIC!
- If SOME constr. are integrable, we say the sys. is PARTIALLY NONHOLONOMIC!!

Holonomic → pos. are constr!
Nonholonom → only velocities are constr!! } constrains dir. but not pos!

Example:

Pfaffian const: q??



} $q = \begin{bmatrix}x\\y\\\theta\end{bmatrix}$ 3 gcs.

look @ unconstr. sys!

Only need θ as a gc!!

⇒ Try and write the CONSTRAINT on x,y in terms of θ.

INTEGRABLE! HOLONOMIC!! { $x = l\cos\theta$ ⇒ $x - l\cos\theta = 0$; $h_1(q) = 0$
{ $y = l\sin\theta$ ⇒ $y - l\sin\theta = 0$; $h_2(q) = 0$

⇒ # of gcs we need = # of unconstr. gcs − # of holonom. constr.!
⇒ HOLONOM. CONSTR → Reduce # of gcs!!

Solve for x,y in terms of θ! { $x - l\cos\theta = 0$, $y - l\sin\theta = 0$; $q = \begin{bmatrix}x\\y\\\theta\end{bmatrix}$ ⇒ 3 eqs. of motion!!

EQUIV CONTROL SYS:

$\omega_1(q)\,\dot{q} = 0 \quad \dots \quad \omega_k(q)\,\dot{q} = 0$ } k independent constr!!

⇒ Rewrite sys. of constr. in MATRIX form:

$$A(q)\,\dot{q} = 0$$

$$\begin{bmatrix} - \omega_1(q) - \\ - \omega_2(q) - \\ \vdots \\ - \omega_k(q) - \end{bmatrix}\dot{q} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

⇒ If $\dot{q}$ satisfies the constr, what do we know about it??
⇒ $\dot{q}$ MUST be in the NULL SPACE of $A(q)$!!
⇒ $q \in \mathbb{R}^n$, k indp. constr: $n - k = m$ dimn. null sp!!

BASIS: $\{g_1(q), \dots, g_m(q)\}$

☆ $\boxed{\dot{q} = u_1 g_1(q) + \dots + u_m g_m(q)}$ $u_i \in \mathbb{R}$ ☆

EQUIV. CONTROL SYSTEM!

A traj. $q(t)$ is ALLOWABLE if $\omega_i(q)\,\dot{q} = 0$ for all t!!

LIE BRACKETS, Controllability:

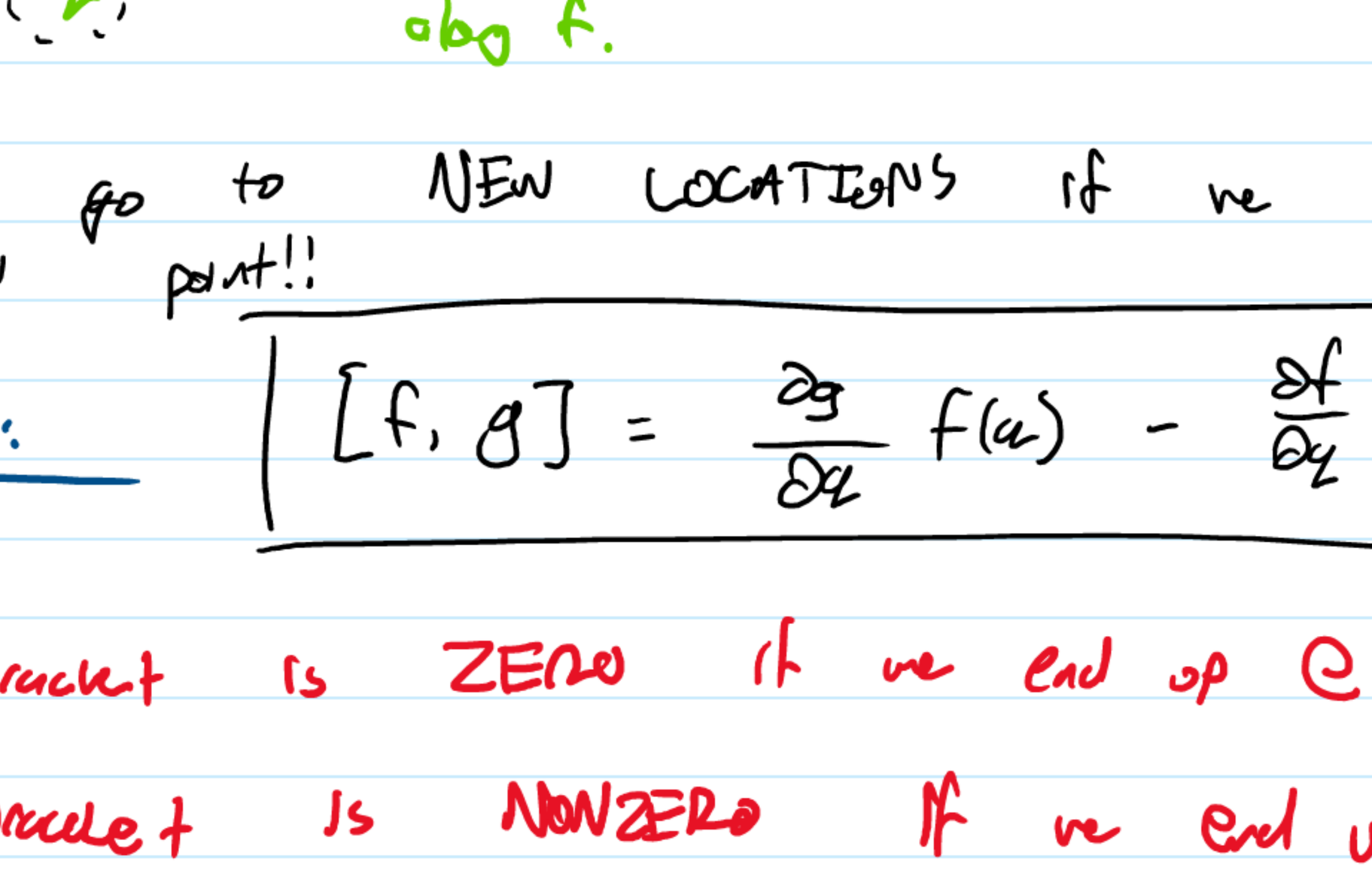
Q: $\dot{q} = u_1 g_1(q) + \dots + u_m g_m(q)$
⇒ Can we move q to ANY DESIRED STATE q_d while satisfying our constraint?? } Controllability!

⇒ ALL MOTIONS of $\dot{q}$ are along VECTOR FIELDS $g_1(q) \dots g_m(q)$
⇒ Vector field: take in a vec, return a vec!!
⇒ When we move along these VFs, can we travel anywhere??

LIE BRACKET:

⇒ Suppose we have TWO VFs, f & g!
⇒ Can travel ANYWHERE by moving along f, g!

$\dot{q} = f(q)$



⇒ Might end up in a diff location!!

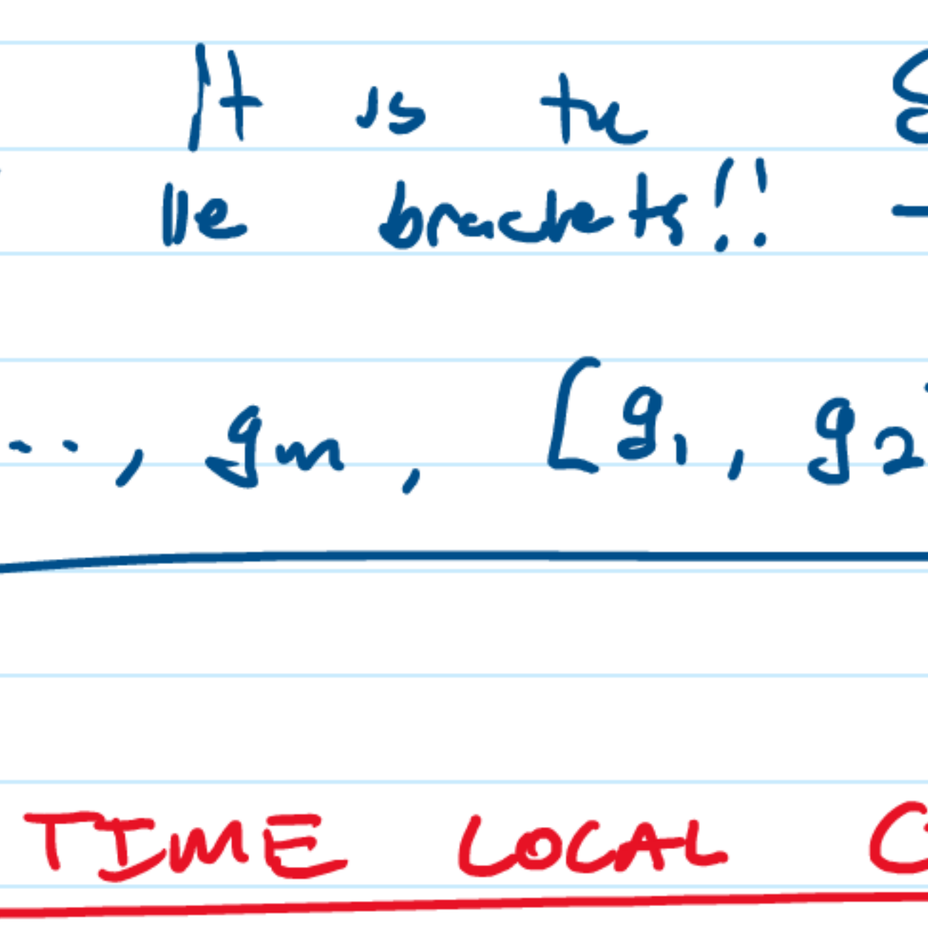
⇒ Can go to NEW LOCATIONS if we end up @ a new point!!

Lie Bracket:

$$\boxed{[f, g] = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q)}$$

} Will have to check @ EACH q!!

⇒ lie bracket is ZERO if we end up @ same place!
- lie bracket is NONZERO if we end up @ a NEW location!!



LIE ALGEBRA: It is the SPAN of VFs. $g_1, \dots, g_m$ & their lie brackets!!

$$g_1, \dots, g_m, [g_1, g_2], \dots, [g_1, [g_2, g_3]] \dots$$

$q_1, \dots, q_n$ } "positions"

Thm: SMALL TIME LOCAL CTRB:

☆ If the lie algebra of $g_1, \dots, g_m$ has dimn. (n) and we can control our inputs $u_1, \dots, u_m$ ARBITRARILY, then at a point q, we can reach ALL points in a ball containing q in finite time!! ☆