# Disc 201.2

Wednesday, February 1, 2023 1:01 PM

## 106B Discussion: 2/02

Today's Agenda:

1. SISO Feedback linearization
2. Relative Degree
3. MIMO feedback linearization (if time :))

## SISO FEEDBACK LINEARIZATION:

General NL System: $\dot{x} = f(x,u)$ $\quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$ } STATE EQN!!

$y = h(x)$ $\quad y \in \mathbb{R}^p$ } OUTPUT EQN!

$y \to$ Contains values we care about!

Control Affine System: $\dot{x} = f(x) + g(x)u$

$y = h(x)$

LINEAR SYSTEM: $\dot{x} = Ax + Bu$

$y = Cx$ } ideal case!!

$\Rightarrow$ Most robots are NONLINEAR.

**Big Idea:** TRANSFORM a nonlinear sys. INTO a linear sys. by Choosing an input u!!

☆ "Feedback linearization" ☆

- Start with the class of Single input, Single output (SISO) Control affine systems!

$$☆ \quad \dot{x} = f(x) + g(x)u \quad ☆ \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}$$
$$y = h(x) \qquad y \in \mathbb{R}$$

$\Rightarrow$ Simplify Problem: focus on $\textcircled{y}$!!

- How the output, y, changes in time!!

$\Rightarrow$ Find a diff. eqn. for y subject to $\dot{x} = f(x) + g(x)u$

$$\dot{y} = \dot{h}(x) = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u\right)$$

$$☆ \quad \dot{y} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)\, u \quad ☆$$

"Lie Derivative" (Lee)

**Defn:** the lie deriv. of $h(x)$ along $f(x)$ is:

$$\boxed{L_f h(x) = \frac{\partial h}{\partial x} f(x)}$$

Components of $L_f h(x)$:

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix} \quad \Big\} \text{ gradient of } h$$

$$f(x) : \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad \Bigg\} \begin{array}{l} \text{vector FIELD!!} \\ \Rightarrow \text{vector-valued func!} \end{array}$$

$$L_f h(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

"DOT PROD / INNER PROD"
$a^T b$ ⇒ how much $a$ goes along $b$

⇒ $L_f h(x)$ is "How QUICKLY" $h(x)$ changes along $f(x)$.
$\frac{\partial h}{\partial x}$

"AWESOME" - Jackson

$$\dot{y} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u$$

$$★ \quad \boxed{\dot{y} = L_f h(x) + L_g h(x) u} \quad ★$$

⇒ $u$ is EASILY separable!!

**Problem 1:** Consider the SISO system above. Assume $L_g h(x) \neq 0$, find a $u$ s.t. when we plug $u$ into $\dot{y} = L_f h + L_g h u$, we get:

$$\boxed{\dot{y} = v}$$

$v \in \mathbb{R}$, $u(v)$...

**Soln:**

$$\boxed{u = \frac{1}{L_g h(x)} \left( -L_f h(x) + v \right)}$$

$$\dot{y} = L_f h + \cancel{L_g h \frac{1}{L_g h}} \left( -L_f h + v \right)$$

$$\dot{y} = \cancel{L_f h - L_f h} + v$$

$$★ \quad \boxed{\dot{y} = v}$$

★ holds for ALL $v$!! Choose $v$ using methods of linear control!!

# High Relative Degree Systems:

$L_g h \neq 0$ } Assumption! $\qquad \dot{y} = L_f h + \cancel{L_g h\, u}$

$\dot{y} = L_f h$ } No Input!!

If $L_g h = 0$, are we stuck??

Solution: take ANOTHER deriv!!

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

} Input term is down here!!

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2 \qquad L_g h = 0 \text{!!}$$

$$\ddot{y} = \dot{x}_2 = x_1 + x_2 + u$$

The RELATIVE DEGREE ($r$) is the NUMBER of derivs. of $y$ we must take for an input to appear.

$$\ddot{y} \Rightarrow r = 2$$

$$\dddot{y} \Rightarrow r = 3$$

$$\vdots$$

Q: How can we take HIGHER derivs. w/ Lie deriv. notation??

$$\dot{y} = L_f h + \cancel{L_g h}\, u^{0}$$

$$\dot{y} = \underline{L_f h(x)}$$

$$\ddot{y} = \frac{\partial}{\partial x}\left(L_f h(x)\right)\left(f(x) + g(x)u\right)$$

$$\ddot{y} = \frac{\partial}{\partial x}\left(L_f h(x)\right) f(x) + \frac{\partial}{\partial x}\left(L_f h(x)\right) g(x)\, u$$

$$= L_f\left[L_f h(x)\right] + L_g\left[L_f h(x)\right] u$$

$$\boxed{\ddot{y} = L_f^2 h(x) + L_g L_f h(x)\, u}$$

## Problem 2: A simple pendulum w/ friction & an input torque:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{\beta}{l} x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = x_1$$

⇒ Take derivs. of y UNTIL an input appears! How many did it take?

Soln: $\dot y = \dot x_1 = x_2$

$$\ddot y = \dot x_2 = -\frac{g}{l}\sin x_1 - \frac{\beta}{l}x_2 + u$$

✓ rel. degree = 2

Problem 3: Now that we have $\ddot y = -\frac{g}{l}\sin x_1 - \frac{\beta}{l}x_2 + u$, find u s.t.

$$\boxed{\ddot y = v}$$

$v \in \mathbb{R}$

☆ $u = \frac{g}{l}\sin x_1 + \frac{\beta}{l}x_2 + v \Rightarrow \boxed{\ddot y = v}$ ☆

## MIMO LINEARIZATION:

- MULTI INPUT, MULTI OUTPUT systems! (MIMO)
  ⇒ "SQUARE" ⇒ number of inputs = number of outputs!!

$$y \in \mathbb{R}^m \Rightarrow u \in \mathbb{R}^m$$

- Want to choose u to lin:

$$\dot x = f(x) + g(x)u \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$
$$y = h(x), \qquad y \in \mathbb{R}^m$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_m(x) \end{bmatrix} \Rightarrow y_j = h_j(x)$$

} "collection" of single input sys.

g(x) is now a matrix! $\in \mathbb{R}^{n \times m}$

$$g(x) = \begin{bmatrix} | & | & & | \\ g_1(x) & g_2(x) & \dots & g_m(x) \\ | & | & & | \end{bmatrix}$$

Problem 4: Show that $\dot y_j$ is computed:

$$\dot y_j = L_f h_j(x) + \sum_{i=1}^{m} L_{g_i} h_j(x) u_i$$

Hint: $\frac{\partial h_j}{\partial x}$ ⇒ try mult. by all columns of g(x)

Soln:

$$\dot{y}_j = \dot{h}_j(x) = \frac{\partial h_j}{\partial x}\left(f(x) + g(x)u\right)$$

$$= L_f h_j(x) + \frac{\partial h_j}{\partial x} g(x)\, u$$

$$= L_f h_j(x) + \begin{bmatrix} \frac{\partial h_j}{\partial x} g_1(x) & \cdots & \frac{\partial h_j}{\partial x} g_m(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$= L_f h_j(x) + \frac{\partial h_j}{\partial x} g_1 u_1 + \cdots + \frac{\partial h_j}{\partial x} g_m u_m$$

$$= L_f h_j(x) + L_{g_1} h_j u_1 + \cdots + L_{g_m} h_j u_m$$

$$\underline{\dot{y}_j} = \boxed{L_f h_j + \sum_{i=1}^{m} L_{g_i} h_j u_i}$$