Discussion 1/25:

Today's Agenda:
1) Defn. of Lyapunov Stability
2) Energy-like funcs.
3) Lyapunov's Basic thm.

LYAPUNOV STABILITY:

⇒ If we have: $\dot{x} = f(x,u)$ (x_e, u_e) is an EQ. PT. IF

$$0 = f(x_e, u_e)$$

- we said: "START CLOSE" ⇒ "STAY CLOSE" = STABLE
- How to involve math?

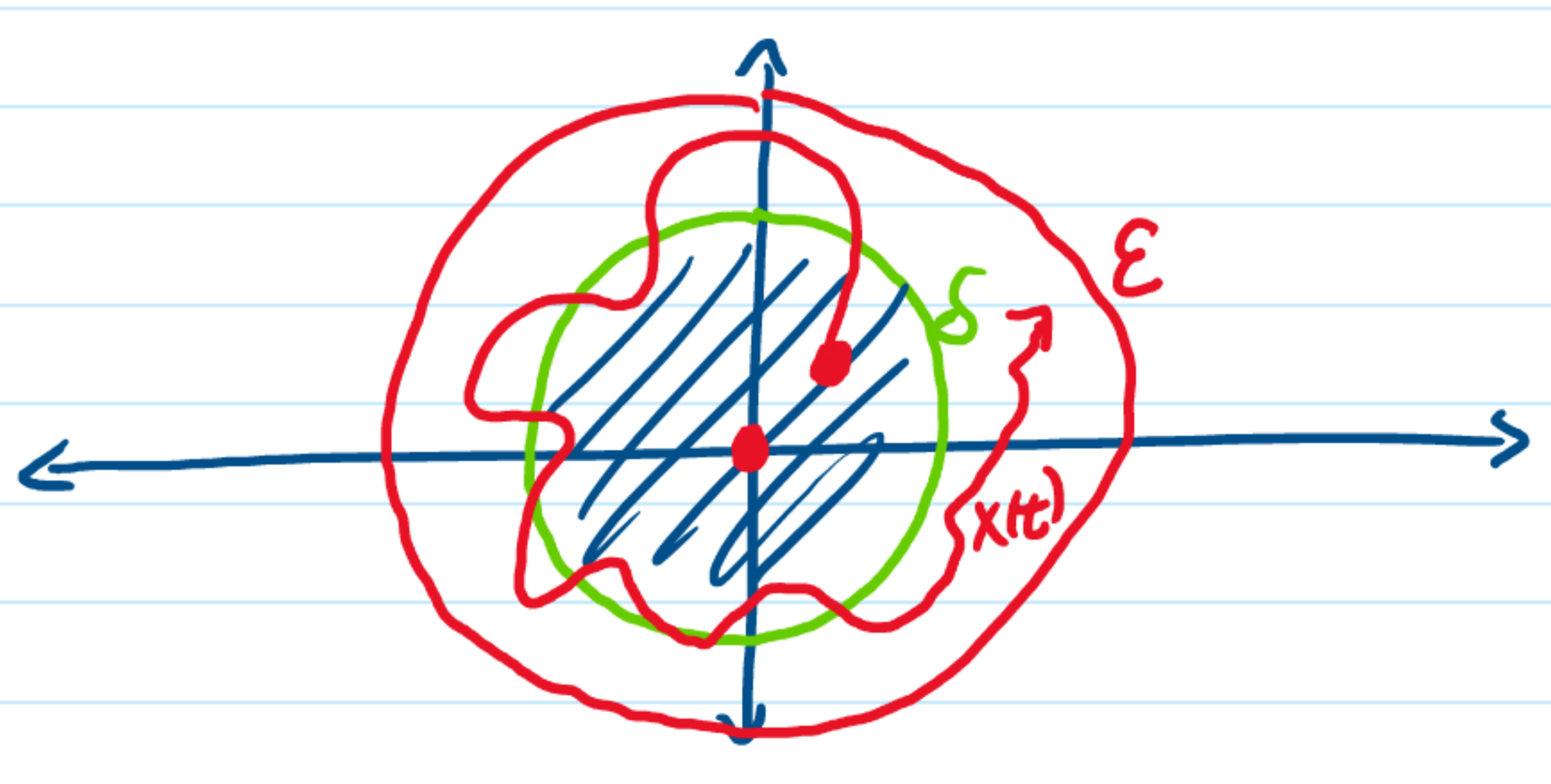
$$\dot{x} = f(x,t)$$ } ignore input for now!!

Assumption: $x_e = 0$ ☆

DEFN: LYAPUNOV STABILITY:

- The point $x_e = 0$ is STABLE IN THE SENSE OF LYAPUNOV (SISL) at $t = t_0$, IF for all $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_0)$ s.t.

if $\|x_0\| < \delta$ THEN: $\|x(t)\| < \varepsilon$ for all $t \geq t_0$



"Start Close" ⇒ $\|x_0\| < \delta$
"STAY CLOSE" ⇒ $\|x(t)\| < \varepsilon$

Given ANY ε, can find a δ!

☆ - SISL ≠ Convergence!! ☆

ENERGY-LIKE FUNCS:

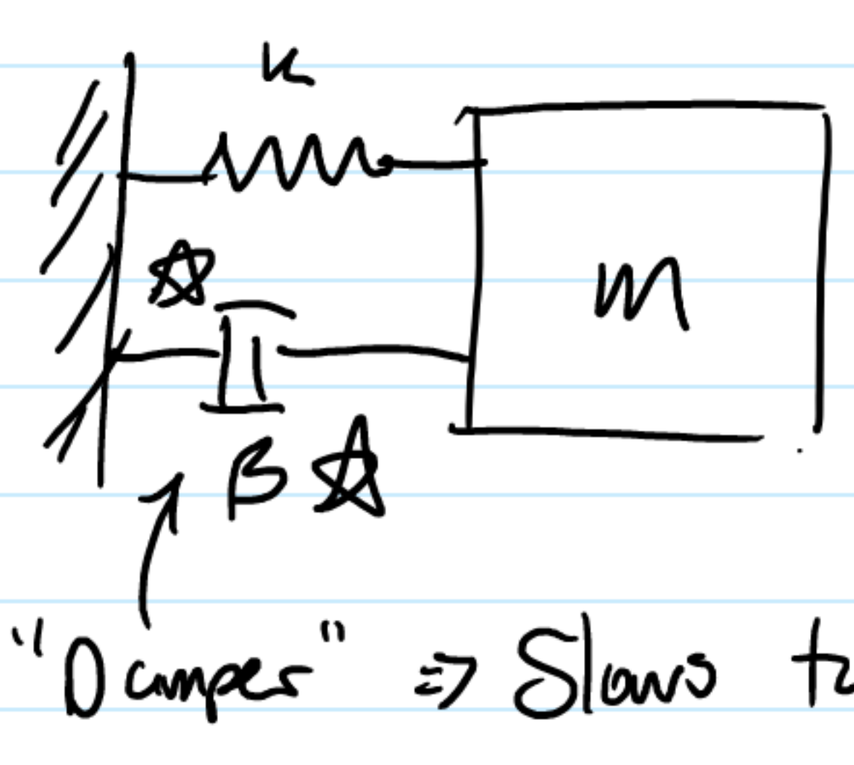
- Lyap. Stability ⇒ BOUND on $\|x(t)\|$
  - to apply defn. DIRECTLY, NEED $x(t)$!!

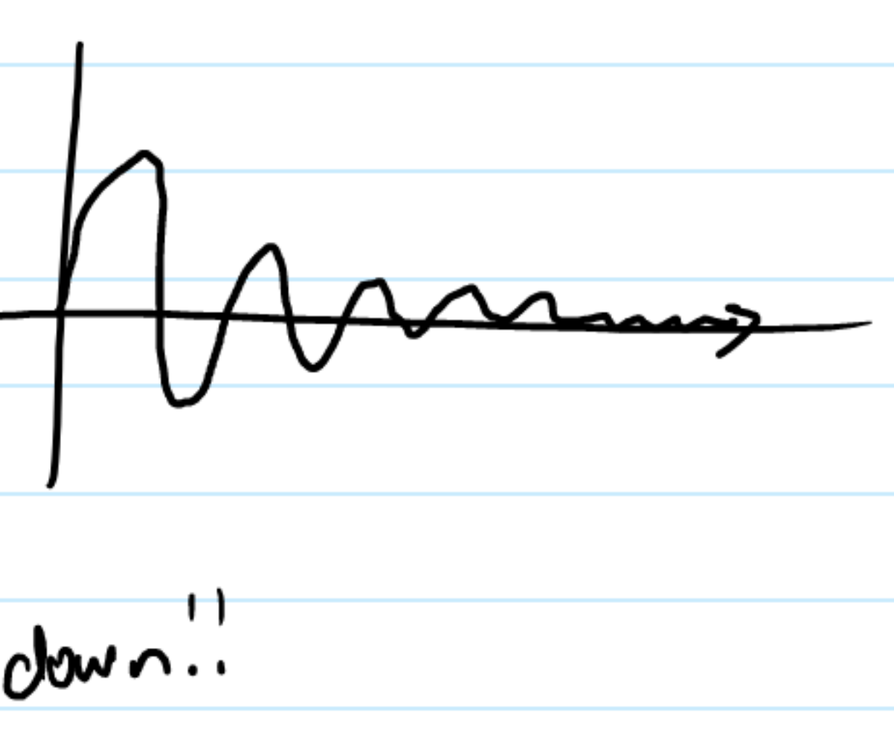
$$\dot{x} = f(x,t)$$

⇒ typically CANNOT find $x(t)$!!

- Descr. Stability WITHOUT using $x(t)$!!

Consider the System:





"Damper" ⇒ Slows the mass down!!

- Damper "BURNS UP ENERGY"
  - Energy drops to 0, system "freezes" ⇒ it goes to its EQUI. PT!!
- If energy is ALWAYS decr:

$$\dot{E} \leq 0$$ for all time!!

⇒ got STABILITY ⇒ we stayed close to our eq. pt!!

- Didn't need to solve ⇒ Just looked at energy!!

IDEA: Use "ENERGY" to study stability of nonlinear systems!!

Problems: 1) might not know how far away we'll go
2) Non-conservative forces

⇒ An arbitrary sys. $\dot{x} = f(x,t)$ is NOT NEC. PHYSICAL!

$$\dot{x} = \sin x$$ } What does energy mean??

- Come up with a set of funcs. that ACT LIKE ENERGY but for an arbitrary sys!!

LYAPUNOV FUNC: $V(x,t)$

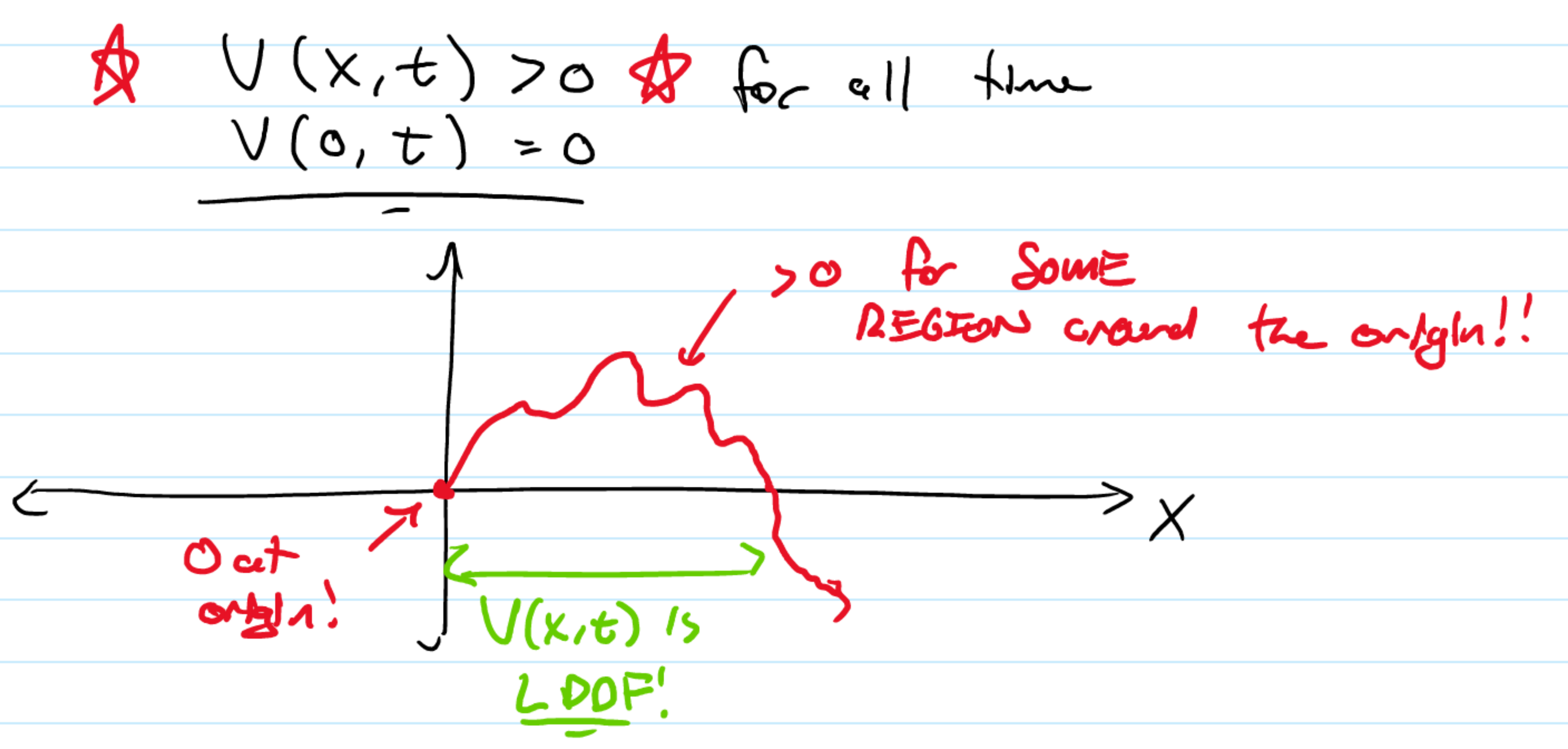
"energy-like func. for NL system!"

TWO POSSIBILITIES / CLASSES for $V(x,t)$

LOCALLY POSITIVE DEFINITE FUNC: (LPDF)

→ $V(x,t)$ is LPDF IF for SOME region around $x = 0$:

☆ $V(x,t) > 0$ ☆ for all time
$V(0,t) = 0$

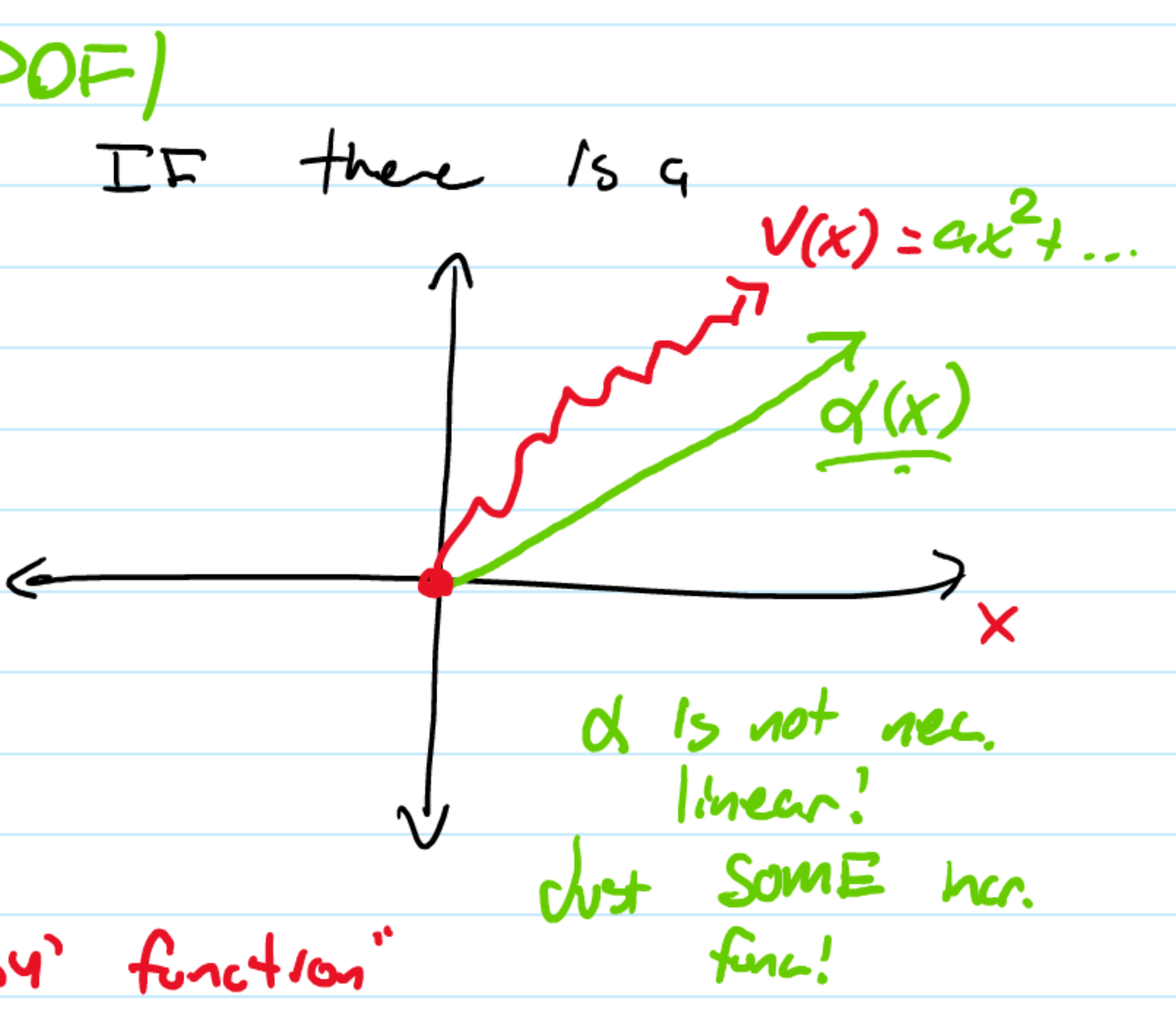


⇒ Minimum at 0!
⇒ >0 in general!! } ~ similar to energy!!

POSITIVE DEFINITE FUNCTION (PDF)

→ $V(x,t)$ is a POSITIVE DEF. FUNC. IF there is a strictly ↑ func. α s.t.

1) $V(0,t) = 0$
2) for all x: $V(x,t) \geq \alpha(\|x\|)$
3) $\lim_{p \to \infty} \alpha(p) = \infty$



"STRICTER FORM of an 'ENERGY' function"

☆ **Problem 1:** let $P \in \mathbb{R}^{n \times n}$ be diagonal & have ALL POSITIVE, REAL eigenvals. ☆

$$P = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Show that $V(x) = x^T P x$, $x \in \mathbb{R}^n$ is a PDF.

Solution:

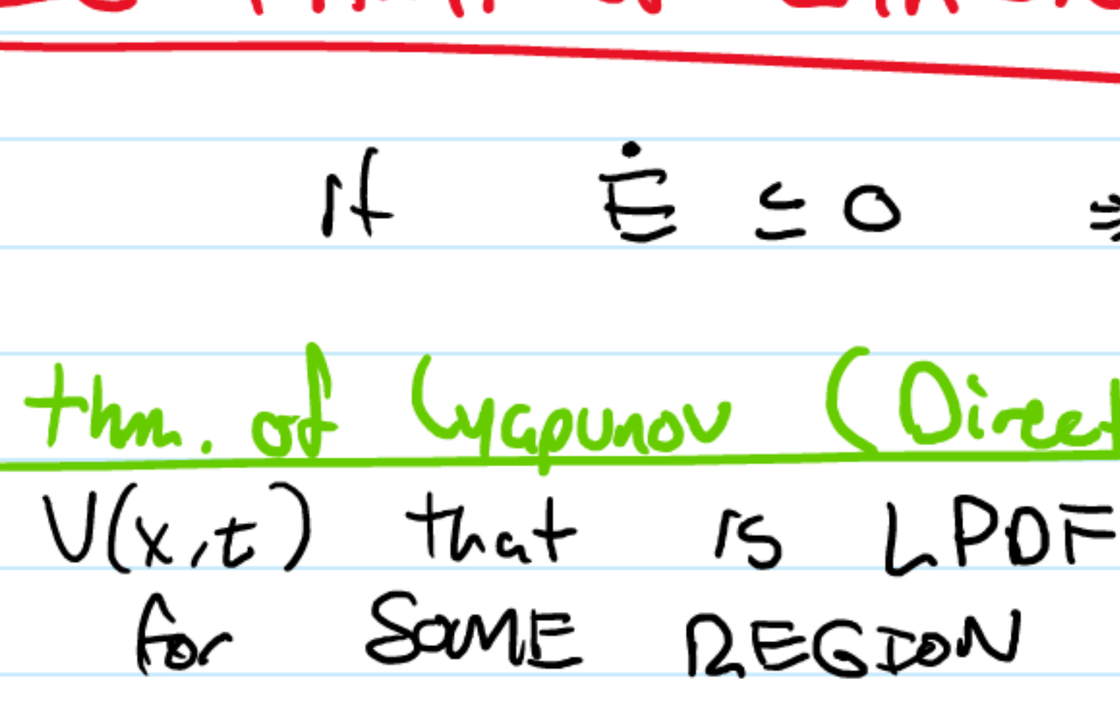
2) $x^T P x \geq \alpha(\|x\|)$ for some α.

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2 \geq \lambda_{min}(x_1^2 + x_2^2 + \dots + x_n^2)$$

$\lambda_{min} = \min\{\lambda_1 \dots \lambda_n\}$ $= \lambda_{min}\|x\|^2 = \alpha(\|x\|)$

↑ STRICTLY INCR!

$\lambda_{min} > 0$



BASIC THM. of LYAPUNOV:

If $\dot{E} \leq 0$ ⇒ maybe conclude stability??

Basic thm. of Lyapunov (Direct Method):

- let $V(x,t)$ that is LPDF
- If, for SOME REGION around the origin, $(x_e = 0)$, we have:

$$\dot{V}(x,t) \leq 0$$

☆ ⇒ Subject to $\dot{x} = f(x,t)$ ☆

Then we conclude $x_e = 0$ is a LOCALLY STABLE EQUI. PT. of $\dot{x} = f(x,t)$

⇒ Uses "energy" idea to conclude stability WITHOUT knowing $x(t)$!!!

Remark: "Derivative along a traj"

- When we take a derivative & USE $\dot{x} = f(x,t)$ as a constr.!!

$$\frac{d\, V(x,t)}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\underbrace{\frac{dx}{dt}}_{\dot{x}}$$ } Chain rule

☆ $$= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t)$$ } ☆ Derivative of V ALONG TRAJ. of $\dot{x} = f(x,t)$

Problem: ☆ $\ddot{\theta} = -a\sin\theta - \beta\dot{\theta}$ ☆ If you get $\ddot{\theta}$, subst. this!!

$$V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + a(1 - \cos\theta)$$ {

Show that for SOME values of $\theta, \dot{\theta}$: $\dot{V}(\theta, \dot{\theta})$ along the trajs. of the sys. is ≤ 0.

$$\dot{V} = \dot{\theta}\ddot{\theta} + a(\sin\theta\, \dot{\theta}) = \dot{\theta}(-a\sin\theta - \beta\dot{\theta}) + a\dot{\theta}\sin\theta$$

↑ Subst. dyn!

$$= -\beta\dot{\theta}^2 \qquad \beta > 0$$

This is ≤ 0 for SOME region! $x_e = 0$ is locally stable!!