118 disc

Wednesday, January 18, 2023 4:09 PM

Disc. 1/18

Agenda:

1) Disc. logistics
2) Dynamical sys.
3) linear control!

Dynamical Systems:

- Sys. whose state evolves w/ time!
- language of dynamics: diff. eqns:

$$m\ddot{x} = f_{in} - mg$$

$\ddot{x} = \frac{d^2x}{dt^2}$

- Want to GENERALIZE:

$$(\ddot{x})^3 = \sin(\ddot{x}^2) + \cos(x)$$

- SOME CONVENTION to descr. ANY nonlinear diff. eqn!!

STATE SPACE REPR:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

STATE EQN.
OUTPUT EQN.

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

1) STATE VECTOR: $x \in \mathbb{R}^n$
   - Smallest collection of vars. that allows us to descr. our sys. @ any pt.
   - Cannot directly control!

2) Input vector: $u \in \mathbb{R}^m$
   - Set of all vars we CAN control!

3) Output vector: $y \in \mathbb{R}^p$
   - State vars. we "care about" (usually)

Problem: $\dot{x} = f(x,u)$ } this is a first order (system) of ODEs!!

A: Need a method to conv. high order sys. to a sys. of 1st order ODEs!

Want to conv: $x^{(n)} = \frac{d^n x}{dt^n} = f(x,u)$ $x \in \mathbb{R}$

→ Phase Variables! Convert n^{th} order → sys. of n 1st order!!

$q_0 = x$
$q_1 = \dot{x}$
$q_2 = \ddot{x}$
$\vdots$
$q_{n-1} = x^{(n-1)}$

$$q = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{bmatrix}$$

$\dot{q} = f(q,u)$

Take deriv. of q:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \vdots \\ \dot{q}_{n-1} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ f(q_0, u) \end{bmatrix}$$

⇒ n 1st order ODEs w/ SAME behavior!!

$\dot{q}_0 = \dot{x} = q_1$
$\dot{q}_1 = \ddot{x} = q_2$
$\vdots$
$\dot{q}_{n-1} = x^{(n)} = f(x,u) = f(q_0,u)$

$$\int \dot{q}\, dt = \int f(q,u)\, dt$$
$$= q_0 + \int f(q,u)\, dt \quad \Big\{ \quad q(t) = \begin{bmatrix} q_0(t) \\ \vdots \\ q_{n-1}(t) \end{bmatrix} \quad x(t)$$

P1:

$$m\ddot{y} = -F\sin\theta$$
$$m\ddot{z} = F\cos\theta - mg$$
$$I\ddot{\theta} = M$$

All are scalars!

⇒ Can you convert this sys. into S.S. form? Find the state eqn. $\dot{q} = f(q,u)$

⇒ Try finding phase vars for EACH ODE! Combine all into q!

let's try it!

What's the state vec. q?

$$q = \begin{bmatrix} y \\ z \\ \theta \\ \dot{y} \\ \dot{z} \\ \dot{\theta} \end{bmatrix} \qquad \dot{q} = \begin{bmatrix} \dot{y} \\ \vdots \\ \dot{\theta} \\ \frac{-F}{m}\sin\theta \\ \vdots \end{bmatrix}$$

$Y = h(q,u)$
$Y = \theta$
$h(q,u) = \theta$

LINEAR ODEs:

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n \times n}$$

- If we have an init. cond. $X(0) = X_0$ the soln. is:

$$X(t) = e^{At} X_0$$

"MATRIX EXP.": $e^{At} = I + At + \frac{(At)^2}{2!} + \dots + \frac{(At)^n}{n!} + \dots$

Problem 2: $A \in \mathbb{R}^{n \times n}$ that is diagonal.

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

λ_i is an eigenval. of A!!

Prove that $e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$.

Proof:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \dots$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & \\ & \ddots & \\ & & \lambda_n t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (\lambda_1 t)^2 & & 0 \\ & \ddots & \\ 0 & & (\lambda_n t)^2 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} \left(1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \dots\right) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & & & \end{bmatrix}$$

→ what is this??

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

□

STABILITY:

- EQUI. PT: $\dot{x} = f(x,u)$ (x_e, u_e) s.t.

$0 = f(x_e, u_e)$ } sys. is "frozen" at an eq. pt.!!

What does it mean for x_e to be STABLE??

$\dot{x} = f(x)$ x_e → $0 = f(x_e)$

"Start Near, Stay Near" ⇒ Stable point.

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}$$

$$\int_0^t \dot{q}\, dt = \int_0^t f(x,u)\, dt$$

$$q(t) - q(0) = \int_0^t f(x,u)\, dt$$