Lecture 5
Uncalibrated Geometry & Stratification
Uncalibrated Camera

\[ x' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K x = \begin{bmatrix} f s_x & f s_\theta & o_x \\ 0 & f s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \]

Linear transformation $K$

calibrated coordinates

pixel coordinates

$(0, 0)$

$(x', y')$

$(u_x, o_y)$

$s_y$

$s_x$
Overview

- Calibration with a rig
- Uncalibrated epipolar geometry
- Ambiguities in image formation
- Stratified reconstruction
- Autocalibration with partial scene knowledge
Uncalibrated Camera

\[ X = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1) \]

Calibrated camera
• Image plane coordinates \( x = [x, y, 1]^T \)
• Camera extrinsic parameters \( g = (R, T) \)
• Perspective projection \( \lambda x = [R, T]X \)

Uncalibrated camera
• Pixel coordinates \( x' = Kx \)
• Projection matrix \( \lambda x' = \Pi X = [KR, KT]X \)
Taxonomy on Uncalibrated Reconstruction

\[ \lambda x' = [KR, KT]X \]

• \( K \) is known, back to calibrated case \( x = K^{-1}x' \)

• \( K \) is unknown
  - Calibration with complete scene knowledge (a rig) – estimate \( K \)
  - Uncalibrated reconstruction despite the lack of knowledge of \( K \)
  - Autocalibration (recover \( K \) from uncalibrated images)

• Use partial knowledge \( K \)
  - Parallel lines, vanishing points, planar motion, constant intrinsic

• Ambiguities, stratification (multiple views)
Uncalibrated Epipolar Geometry

\[ \lambda_2 K x_2 = KR \lambda_1 x_1 + KT \]

\[ \lambda_2 x'_2 = KRK^{-1} \lambda_1 x'_1 + T' \]

- Epipolar constraint
  \[ x'^T \underbrace{K^{-T} \hat{R} K^{-1}}_{F} x'_1 = 0 \]

- Fundamental matrix
  \[ F = K^{-T} \hat{R} K^{-1} \]

- Equivalent forms of
  \[ F = K^{-T} \hat{R} K^{-1} = \hat{T}' K R K^{-1} \]
Properties of the Fundamental Matrix

\[ x'_1^T F x'_1 = 0 \]

- Epipolar lines \( l_1, l_2 \)
- Epipoles \( e_1, e_2 \)

\[ l_1 \sim F^T x'_2 \]
\[ F e_1 = 0 \]
\[ l_i^T x'_i = 0 \]
\[ l_i^T e_i = 0 \]
\[ l_2 \sim F x'_1 \]
\[ e_2^T F = 0 \]
Properties of the Fundamental Matrix

\[ x'^T_2 F x'_1 = 0 \]

- Epipolar lines \( l_1, l_2 \)
- Epipoles \( e_1, e_2 \)

\[ l_1 \sim F^T x'_2 \]
\[ F e_1 = 0 \]

\[ l_i^T x'_i = 0 \]
\[ l_i^T e_i = 0 \]
\[ l_2 \sim F x'_1 \]
\[ e_2^T F = 0 \]
Properties of the Fundamental Matrix

A nonzero matrix $F \in \mathbb{R}^{3 \times 3}$ is a fundamental matrix if $F$ has a singular value decomposition (SVD) $F = U \Sigma V^T$ with

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}^+_\text{.}$

There is little structure in the matrix $F$ except that

$$\det(F) = 0$$
Estimating Fundamental Matrix

- Find such F that the epipolar error is minimized
  \[ \min_F \sum_{j=1}^{n} x_2^j F x_1^j \quad \text{Pixel coordinates} \]

- Fundamental matrix can be estimated up to scale

- Denote \( a = x_1' \otimes x_2' \)
  \[ a = [x_1 x_2, x_1 y_2, x_1 z_2, y_1 x_2, y_1 y_2, y_1 z_2, z_1 x_2, z_1 y_2, z_1 z_2]^T \]
  \[ F^s = [f_1, f_4, f_7, f_2, f_5, f_8, f_3, f_6, f_9]^T \]

- Rewrite
  \[ a^T F^s = 0 \]

- Collect constraints from all points
  \[ \chi F^s = 0 \]

  \[ \min_F \sum_{j=1}^{n} x_2^j F x_1^j \quad \rightarrow \quad \min_{F^s} \| \chi F^s \|^2 \]
Two view linear algorithm – 8-point algorithm

- Solve the LLSE problem:

\[ \min_F \sum_{j=1}^{n} x_2'^j F x_1'^j \Rightarrow \chi F^s = 0 \]

- Solution eigenvector associated with smallest eigenvalue of \( \chi^T \chi \)

- Compute SVD of F recovered from data

\[ F = U \Sigma V^T \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3) \]

- Project onto the essential manifold:

\[ \Sigma' = \text{diag}(\sigma_1, \sigma_2, 0) \quad F = U \Sigma' V^T \]

- \( F \) cannot be unambiguously decomposed into pose and calibration

\[ F = K^{-T} \hat{\Theta} R K^{-1} \]

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Figure 6.1. Effect of the matrix $K$ as a map $K : v \rightarrow u = K v$, where points on the sphere $\|v\|^2 = 1$ is mapped to points on an ellipsoid $\|u\|_S^2 = 1$ (a “unit sphere” under the metric $S$). Principal axes of the ellipsoid are exactly the eigenvalues of $S$. 
Calibrated vs. Uncalibrated Space

Distances and angles are modified by $S$
Motion in the distorted space

\[ X(t) = R(t)X(t_0) + T(t) \quad \text{Calibrated space} \]

\[ KX(t) = KR(t)X(t_0) + KT(t) \quad \text{Uncalibrated space} \]

\[ X(t) = R(t)X(t_0) + T(t) \quad X'(t) = KR(t)K^{-1}X'(t_0) + KT(t) \]

- Uncalibrated coordinates are related by

\[ G' = \left\{ g' = \begin{bmatrix} KRK^{-1} & T' \\ 0 & 1 \end{bmatrix} \mid T' \in \mathbb{R}^3, R \in SO(3) \right\} \]

- Conjugate of the Euclidean group
What Does $F$ Tell Us?

- $F$ can be inferred from point matches (eight-point algorithm)
- Cannot extract motion, structure and calibration from one fundamental matrix (two views)
- $F$ allows reconstruction up to a projective transformation (as we will see soon)
- $F$ encodes all the geometric information among two views when no additional information is available
Decomposing the Fundamental Matrix

\[ F = K^{-T} \hat{T} RK^{-1} = \hat{T}' KR K^{-1} \]

- Decomposition of the fundamental matrix into a skew symmetric matrix and a nonsingular matrix
  \[ F \leftrightarrow \Pi = [R', T'] \implies F = \hat{T}' R'. \]

- Decomposition of \( F \) is not unique
  \[ x_2' \hat{T}' (T'v^T + KR K^{-1}) x_1' = 0 \quad T' = KT \]

- Unknown parameters - ambiguity
  \[ v = [v_1, v_2, v_3]^T \in \mathbb{R}^3, \quad v_4 \in \mathbb{R} \]

- Corresponding projection matrix
  \[ \Pi = [KR K^{-1} + T'v^T, v_4 T'] \]
Projective Reconstruction

- From points, extract $F'$, followed by computation of projection matrices $\Pi_{ip}$ and structure $X_p$
- Canonical decomposition
  
  $$F \mapsto \Pi_{1p} = [I, 0], \quad \Pi_{2p} = [(\hat{T}')^T F, T']$$

- Given projection matrices – recover structure $X_p$
  
  $$\lambda_1 x'_1 = \Pi_{1p} X_p = [I, 0] X_p,$$
  $$\lambda_2 x'_2 = \Pi_{2p} X_p = [(\hat{T}')^T F, T'] X_p.$$ 

- Projective ambiguity – non-singular 4x4 matrix $H_p$
  
  $$\lambda_i x'_i = \Pi_{ip} H^{-1} H X_p$$
  $$\lambda_i x'_i = \tilde{\Pi}_{1p} \tilde{X}_p$$

Both $\Pi_{ip}$ and $\tilde{\Pi}_{ip}$ are consistent with the epipolar geometry – give the same fundamental matrix
Projective Reconstruction

• Given projection matrices recover projective structure

\[(x_1 \pi_1^{3T})X_p = \pi_1^T X_p, \quad (y_1 \pi_1^{3T})X_p = \pi_1^2 T X_p,
\]

\[(x_2 \pi_2^{3T})X_p = \pi_2^T X_p, \quad (y_2 \pi_2^{3T})X_p = \pi_2^2 T X_p,\]

• This is a linear problem and can be solve using linear least-squares

\[MX_p = 0\]

• Projective reconstruction – projective camera matrices and projective structure

\[X_e = HX_p\]
Euclidean vs Projective reconstruction

- **Euclidean reconstruction** – true metric properties of objects lengths (distances), angles, parallelism are preserved
  - Unchanged under rigid body transformations
  - $\Rightarrow$ Euclidean Geometry – properties of rigid bodies under rigid body transformations, similarity transformation

- **Projective reconstruction** – lengths, angles, parallelism are **NOT** preserved – we get distorted images of objects – their distorted 3D counterparts --> 3D projective reconstruction
  - $\Rightarrow$ Projective Geometry
Homogeneous Coordinates (RBM)

3-D coordinates are related by:

\[ X_c = RX_w + T, \]

Homogeneous coordinates:

\[ X = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow X = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4, \]

Homogeneous coordinates are related by:

\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c \\
1
\end{bmatrix} =
\begin{bmatrix}
R & T \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
X_w \\
Y_w \\
Z_w \\
1
\end{bmatrix}
\]
Homogenous and Projective Coordinates

- Homogenous coordinates in 3D before – attach 1 as the last coordinate – render the transformation as linear transformation
- Projective coordinates – all points are equivalent up to a scale

\[ X = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \approx X = \begin{bmatrix} WX \\ WY \\ W \end{bmatrix} \in \mathbb{R}^3 \]

2D- projective plane

\[ X = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \approx X = \begin{bmatrix} WX \\ WY \\ WZ \\ W \end{bmatrix} \in \mathbb{R}^4 \]

3D- projective space

- Each point on the plane is represented by a ray in projective space

\[ X = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} \]

\[ X = \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} \in \mathbb{R}^4 \]

- Ideal points – last coordinate is 0 – ray parallel to the image plane
- Points at infinity – never intersects the image plane
Vanishing points – points at infinity

Representation of a 3-D line – in homogeneous coordinates

\[
X = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}, \quad \mu \in \mathbb{R}
\]

When \( \lambda \to 1 \) - vanishing points – last coordinate \( \to 0 \)

\[
X = \begin{bmatrix} X_0 + \lambda V_1 \\ Y_0 + \lambda V_2 \\ Z_0 + \lambda V_3 \\ 1 \end{bmatrix} \quad X = \begin{bmatrix} X_0/\lambda + V_1 \\ Y_0/\lambda + V_2 \\ Z_0/\lambda + V_3 \\ 1/\lambda \end{bmatrix} \quad X = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}
\]

Similarly in the image plane
Ambiguities in the image formation

\[ \lambda x' = K \Pi_0 gX \]

- Potential Ambiguities

\[ \lambda x' = \Pi X = K \Pi_0 gX = KR_0^{-1}R_0 \Pi_0 H^{-1} Hgg_w^{-1} g_w X \]

- Ambiguity in \( K \) (\( K \) can be recovered uniquely – Cholesky or QR)

\[ \lambda x' = K \Pi_0 gX = KR_0R_0^{-1}[R, T]X \overset{\text{\( \hat{K} \) \( \Pi_0 \) \( \hat{g} \) \( X \)}}{=} \]

- Structure of the motion parameters

\[ gX = gg_w^{-1} g_w X \]

- Just an arbitrary choice of reference frame
Ambiguities in Image Formation

Structure of the (uncalibrated) projection matrix \( \Pi = [KR, KT] \)

\[ \lambda x' = \Pi X = (\Pi H^{-1})(HX) = \tilde{\Pi} \tilde{X} \]

- For any invertible 4 x 4 matrix \( H \)

- In the uncalibrated case we cannot distinguish between camera \( \Pi \) imaging word \( X \) from camera \( \tilde{\Pi} \) imaging distorted world \( \tilde{X} \)

- In general, \( H \) is of the following form

\[
H^{-1} = \begin{bmatrix}
G & b \\
0 & 1
\end{bmatrix}
\]

- In order to preserve the choice of the first reference frame we can restrict some DOF of \( H \)
Structure of the Projective Ambiguity

- 1st frame as reference
  \[ \lambda_1 x'_1 = K_1 \Pi_0 X_e \]
  \[ \lambda_1 x'_1 = K_1 \Pi_0 H^{-1} H X_e = \Pi_{1p} X_p \]

- Choose the projective reference frame
  \[ \Pi_{1p} = [I_{3 \times 3}, 0] \] then ambiguity is
  \[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} \]
  \[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \]
  \[ = H_a^{-1} H_p^{-1} \]

\[ X_p = H_p \underbrace{H_a g_e X}_{X_e} \]
Stratified (Euclidean) Reconstruction

- General ambiguity – while preserving choice of first reference frame
  \[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} \]

- Decomposing the ambiguity into affine and projective one
  \[ H^{-1} = H_a^{-1} H_p^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \]

- Note the different effect of the 4-th homogeneous coordinate
Affine upgrade

- Upgrade projective structure to an affine structure

\[
H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \quad X_a = H_p^{-1}X_p
\]

- Exploit partial scene knowledge
  - Vanishing points, no skew, known principal point
- Special motions
  - Pure rotation, pure translation, planar motion, rectilinear motion
- Constant camera parameters (multi-view)
Affine upgrade using vanishing points

How to compute

$$H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \quad X_a = H_p^{-1}X_p$$

Maps the points

$$[v, v_4]^T X_p = 0$$

To points with affine coordinates

$$X_a = [X, Y, Z, 0]^T$$

$$X_a = [X, Y, Z, 0]^T$$

Vanishing points – last homogeneous affine coordinate is 0
Affine Upgrade

Need at least three vanishing points

$$[v, v_4]^T X_p^i = 0, \ i = 1, 2, 3$$

3 equations, 4 unknowns (-1 scale)

Solve for

$$[v, v_4] = [v_1, v_2, v_3, v_4]$$

Set up $H_p^{-1}$ and update the projective structure

$$X_a = H_p^{-1} X_p$$
Euclidean upgrade

- We need to estimate remaining affine ambiguity

\[ H_a^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \]

**Alternatives:**

- In the case of special motions (e.g. pure rotation) – no projective ambiguity – cannot do projective reconstruction

\[ \lambda_2 x_2' = R_a \lambda_1 x_1' \]

\[ R_a = KRK^{-1} \Rightarrow R_a (KKT) R_a^T = (KKT) \]

- Estimate \( KKT \) directly (special case of rotating camera – follows)
- Multi-view case – estimate projective and affine ambiguity together
- Use additional constraints of the scene structure (next)
- Autocalibration (Kruppa equations)
Direct Stratification from Multiple Views

From the recovered projective projection matrix
\[ \pi_{ip} = \pi_{ie} H^{-1} = [B_i, b_i], \quad B_i \in \mathbb{R}^{3 \times 3}, b_i \in \mathbb{R}^3 \]

we obtain the absolute quadric constraints
\[
(B_i - b_i v^T) K K^T (B_i - b_i v^T)^T = \lambda K K^T
\]

Partial knowledge in \( K \) (e.g. zero skew, square pixel) renders the above constraints linear and easier to solve.

The projection matrices can be recovered from the multiple-view rank method to be introduced later.
## Geometric Stratification

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<th>3-D structure</th>
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<td>$\Pi_1e = [K, 0], \ Pi_2e = [KR, KT]$</td>
<td>$X_e = g_eX = \begin{bmatrix} R_e &amp; T_e \ 0 &amp; 1 \end{bmatrix} X$</td>
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<tr>
<td>$\Pi_2a = [KRK^{-1}, KT]$</td>
<td>$X_a = H_aX_e = \begin{bmatrix} K &amp; 0 \ 0 &amp; 1 \end{bmatrix} X_e$</td>
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<tr>
<td>$\Pi_{2p} = [KRK^{-1} + KTv^T, v_4KT]$</td>
<td>$X_p = H_pX_a = \begin{bmatrix} I &amp; 0 \ -v^T &amp; v_4^{-1} \end{bmatrix} X_a$</td>
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![Images of 3D projections](image1.png)  

$X_e$  

$X_a$  

$X_p$
## Overview of the methods

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<td>$(x'_2)^TFx'_1 = 0$</td>
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<td>$X_a \leftarrow X_p$</td>
<td>$X_p \leftarrow {x'_1, x'_2}$</td>
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<td><strong>Info. needed</strong></td>
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<tr>
<td><strong>Info. needed</strong></td>
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Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.
Calibration with a Rig

• Given 3-D coordinates on known object \( \mathbf{X} \)

\[
\lambda \mathbf{x}' = [KR, KT] \mathbf{X} \quad \Rightarrow \quad \lambda \mathbf{x}' = \prod \mathbf{X}
\]

\[
\lambda \begin{bmatrix} x^i \\ y^i \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} \begin{bmatrix} X^i \\ Y^i \\ Z^i \\ 1 \end{bmatrix}
\]

• Eliminate scale, two linear constraints per point:

\[
x^i(\pi_3^T \mathbf{X}) = \pi_1^T \mathbf{X},
\]

\[
y^i(\pi_3^T \mathbf{X}) = \pi_2^T \mathbf{X}
\]

• Recover projection matrix \( \prod = [KR, KT] = [R', T'] \)

\[
\prod^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T
\]

\[
\min \| M \prod^s \|^2 \quad \text{subject to} \quad \| \prod^s \|^2 = 1
\]

• Factor the \( KR \) into \( R \in SO(3) \) and \( K \) using QR decomposition

• Solve for translation \( T = K^{-1} T' \)
Calibration with a Planar Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated plane are known.
Calibration with a Planar Rig

• Special world frame on the plane

\[ \lambda x' = \Pi X = [KR, KT]X \]

\[ X = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} \]

• Homography from the plane to the image

\[ \lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K[r_1, r_2, T] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \]

• Two linear constraints on the calibration \( S = K^{-T}K^{-1} \) per image

\[ H \doteq K[r_1, r_2, T] \in \mathbb{R}^{3 \times 3} \]

\[ K^{-1}[h_1, h_2] \sim [r_1, r_2] \]

\[ h_1^T K^{-T}K^{-1}h_2 = 0, \quad h_1^T K^{-T}K^{-1}h_1 = h_2^T K^{-T}K^{-1}h_2. \]
Calibration with Scene Structure: vanishing points

- Vanishing points – intersections of the parallel lines
  \[ v_i = l_1 \times l_2 = \hat{l}_1 l_2 \]

- Vanishing points of three orthogonal directions
  \[ v_1 = KR e_1, \quad v_2 = KR e_2, \quad v_3 = KR e_1 \]

- Orthogonal directions – inner product is zero
  \[ v_i^T S v_j = v_i^T K^{-T} K^{-1} v_j = e_i^T R^T R e_j = e_i^T e_j = 0, \quad i \neq j, \]

- Provide directly constraints on matrix \( S = K^{-T} K^{-1} \)

- \( S \) – has 5 degrees of freedom, 3 vanishing points gives three linear constraints (need additional assumption on \( K \))
- Assume zero skew and aspect ratio = 1
Calibration with Motions – Pure Rotation

- Calibrated two views related by rotation only
  \[ \lambda_2 x_2 = R \lambda_1 x_1 \]

- Mapping to a reference view – rotation can be estimated
  \[ \widehat{x}_2 R x_1 = 0 \]

- Mapping to a cylindrical surface
Calibration with Motions: Pure Rotation

- Uncalibrated two views related by a pure rotation:
  \[ \lambda_2 K x_2 = \lambda_1 K R K^{-1} K x_1 \quad \widehat{x}_2 K R K^{-1} x'_1 = 0 \]

- Conjugate rotation \( C = K R K^{-1} \) can be estimated

- Given \( C \), we have **three linear constraints**:
  \[ S^{-1} - C S^{-1} C^T = 0 \text{ where } S^{-1} = K K^T \]

- Given **two rotations** around linearly independent axes – \( S, K \) can be estimated using linear techniques

- Applications – image mosaics
Calibration with Motions: General Motions

The fundamental matrix

\[ F = K^{-T} \hat{T}RK^{-1} = \hat{T}'KRK^{-1} \]

does not satisfy the Kruppa’s equations

\[ FKK^TF^T = \hat{T}'KK^T\hat{T}'^T \]

If the fundamental matrix is known up to scale

\[ FKK^TF^T = \chi^2\hat{T}'KK^T\hat{T}'^T \]

This give two nonlinear constraints on \( S^{-1} = KK^T \)

Solution to Kruppa’s equations can be sensitive to noises.
Under special motions,

1. $\omega$ is parallel to $T$ (i.e. the screw motion), and
2. $\omega$ is perpendicular to $T$ (e.g., the planar motion).

The scale $\lambda$ can be determined, hence the Kruppa’s equations become linear in $S^{-1} = KK^T$.

$$FKK^TF^T = \lambda^2 T' K K^T T'^T$$

Each Kruppa equation gives two linear constraints on

$$S^{-1} = KK^T$$
### Calibration with Motions: Special Motions

<table>
<thead>
<tr>
<th>Cases</th>
<th>Type of constraints</th>
<th># of constraints on $S^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 0$</td>
<td>Lyapunov equation (linear)</td>
<td>3</td>
</tr>
<tr>
<td>$R \perp T$</td>
<td>Normalized Kruppa (linear)</td>
<td>2</td>
</tr>
<tr>
<td>$R \parallel T$</td>
<td>Normalized Kruppa (linear)</td>
<td>2</td>
</tr>
<tr>
<td>Others</td>
<td>Unnormalized Kruppa (nonlinear)</td>
<td>2</td>
</tr>
</tbody>
</table>

Invitation to 3D vision